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Cesàro means of integrable functions with respect to unbounded Vilenkin systems ☆

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Abstract

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups. In 1999 the author proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. One of the main difficulties is that the sequence of the L^1 norm of the Fejér kernels is not bounded. This is a sharp contrast between the unbounded and the bounded Vilenkin systems. The aim of this paper is to discuss the L^1 case. We prove for $f \in L^1(G_m)$ that the relation $\sigma_{M_n} f \to f$ holds a.e. (M_n is the nth generalized power).

MSC: 42C10

Keywords: $(C,1)$ means; Vilenkin series; A.e. convergence

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1. Introduction

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups.

Fine [4] proved every Walsh–Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C,α) summable for $\alpha > 1$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [9]. Schipp [12] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. ($f \in L^1(G_m)$). He proved that σ^* is of weak type (L^1, L^1). That σ^* is bounded from H^1 to L^1 was discovered by Fujii [5].

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [15] and later to bounded Vilenkin systems by Pál and Simon [10].

The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [11]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [11] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [11] that if $\frac{\log m_n}{M_n} \to \infty$, then there exist a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable. That is, only, a.e. convergence can be stated for unbounded Vilenkin groups. The almost everywhere convergence of the full partial sums for $L^p, p > 1$, is known in the bounded case [6] but not in the unbounded case. On the other hand, mean convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. Namely, in 1999 Gát [7] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups.

The aim of this paper is to give a partial answer for L^1 case. We discuss a partial sequence of the sequence of the Fejér means. Namely, we prove. Let $f \in L^1(G_m)$. Then we have $\sigma_{M_n} f \to f$ almost everywhere.

For a more complete references we mention the paper of Zheng [17], where the pointwise convergence of Cesàro means of L^1 functions is proved on the setting of local fields. For more general systems see paper [8].

First we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by Vilenkin in 1947 (see e.g. [1,16]) as follows.

Let $m := (m_k,\ k \in \mathbb{N})(\mathbb{N} := \{0, 1, \dots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, with the group operation mod m_k addition. Since the groups is discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k \ (j \in \{0, 1, \dots, m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i,\ i \in \mathbb{N})$, where $x_i \in Z_{m_i}\ (i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinatewise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group.

The Vilenkin group metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by below are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m,\ n \in \mathbb{P} := \mathbb{N} \backslash \{0\}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m, $I_n := I_n(0)\ (n \in \mathbb{N})$.

Furthermore, let $L^p(G_m)\ (1 \leqslant p \leqslant \infty)$ denote the usual Lebesgue spaces ($||.||_p$ the corresponding norms) on G_m, $\mathscr{A}_n$ the σ algebra generated by the sets $I_n(x)\ (x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathscr{A}_n\ (n \in \mathbb{N})$ $(E_{-1}f := 0\ (f \in L^1))$.

The concept of the maximal Hardy space [13] $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|\,(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm

$$||f||_{H^1} := ||f^*||_1.$$

Let X and Y be either $H^1(G_m)$ or $L^p(G_m)$ for some $1 \leqslant p \leqslant \infty$ with norms $||.||_X$ and $||.||_Y$. We say that operator T is of type (Y, X) if there exist an absolute constant $C > 0$ for which $||Tf||_Y \leqslant C||f||_X$ for all $f \in X$. T is of weak type (L^1, L^1) if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \leqslant C||f||_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leqslant \infty$ (see e.g. [2,6]).

Let $M_0 := 1, M_{n+1} := m_n M_n\ (n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \ldots, m_i - 1\}, i \in \mathbb{N}),$$

where only a finite number of n_is differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & \text{if } x_n \neq 0, \\ m_n & \text{if } x_n = 0 \end{cases}$ $(x \in G_m, n \in \mathbb{N})$. The nth Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\operatorname{mod} m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k\oplus n} = \psi_k \psi_n, \psi_n(x+y) = \psi_n(x)\psi_n(y), \psi_n(-x) = \bar{\psi}_n(x), |\psi_n| = 1 (k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f \bar{\psi}_n,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k,$$

$$D_n(y, x) = D_n(y - x) := \sum_{k=0}^{n-1} \psi_k(y) \bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f,$$

$$K_n(y, x) = K_n(y - x) := \frac{1}{n} \sum_{k=1}^{n} D_k(y - x),$$

$$\left(n \in \mathbb{P},\ y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f, S_0 f = D_0 = 0,\ f \in L^1(G_m)\right).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y - x)\, dx,$$

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y - x)\, dx$$

$$(n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0), \\ 0 & \text{if } x \notin I_n(0), \end{cases}$$

$$S_{M_n}f(x) = M_n \int_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Moreover [11] for $n \in \mathbb{N}$,

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{i=m_j-n_j}^{m_j-1} r_j^i.$$

That is, for $z \in I_t \backslash I_{t+1} (t \in \mathbb{N})$

$$D_n(z) = \psi_n(z) \left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(z) \right).$$

2. The theorem

The aim of this paper is to give a partial answer for L^1 case. We discuss a partial sequence of the sequence of the Fejér means. Namely, we prove:

Theorem 2.1. *Let $f \in L^1(G_m)$. Then we have $\sigma_{M_n} f \to f$ almost everywhere.*

Nevertheless, this is only a partial answer. We do not know what to say for the whole sequence of the $(C,1)$ means of integrable functions. On the other hand, we feel that the L^p result [6] and Theorem 2.1 qualify us for giving the conjecture that the theorem of Lebesgue ($\sigma_n f \to f$ a.e.) holds on unbounded Vilenkin groups, too.

In order to prove Theorem 2.1 we need several lemmas. The first one is the so-called Calderon–Zygmund decomposition lemma [3] on unbounded Vilenkin groups (for the proof see e.g. [14]). For $z \in G_m,\ k \in \mathbb{N},\ j \in \{0, \dots, m_k - 1\}$ we use the notation

$$I_k(z,j) = I_{k+1}(z_0, \dots, z_{k-1}, j).$$

Lemma 2.2. *Let $f \in L^1(G_m)$, and $\lambda > ||f||_1 > 0$ arbitrary. Then the function f can be decomposed in the following form*:

$$f = f_0 + \sum_{j=1}^{\infty} f_j, \quad ||f_0||_\infty \leqslant C\lambda, \quad ||f_0||_1 \leqslant C||f||_1,$$

$$\operatorname{supp} f_j \subset \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l) = J_j, \quad \int_{G_m} f_j d\mu = 0 \quad (j \in \mathbb{P}),$$

and for

$$F = \bigcup_{j \in \mathbb{P}} J_j, \quad \mu(F) \leqslant C \frac{\|f\|_1}{\lambda}.$$

Moreover, the sets J_j are disjoint $(j \in \mathbb{P})$.

The second one is as follows. For an integrable function f we define the following operator:

$$H_1 f(y) := \sup_{A \in \mathbb{N}} \left| M_{A-1} \int_{\bigcup_{x_{A-1} \neq y_{A-1}} I_A(y_0, \ldots, y_{A-2}, x_{A-1})} f(x) \frac{1}{1 - r_{A-1}(y-x)} dx \right|.$$

Lemma 2.3. *The operator H_1 is of type (L^2, L^2).*

Proof. Suppose that $a, b, z \in Z_p$, where $2 \leqslant p \in \mathbb{N}$. If $2|p$, then $\cot(\pi \frac{p/2}{p}) = 0$, and $\cot(\pi \frac{z}{p}) = -\cot(\pi \frac{p-z}{p})$ $(z \in \{1, \ldots, p/2 - 1\})$. If $2 \nmid p$, then $\cot(\pi \frac{z}{p}) = -\cot(\pi \frac{p-z}{p})$ for $z \in \{1, \ldots, (p-1)/2\}$. *This gives*

$$\sum_{z=1}^{p-1} \cot\left(\pi \frac{z}{p}\right) = 0.$$

Let $a, b, z \in Z_p$ be different. Then by the above we have

$$\sum_{\substack{z=0 \\ z \neq a,b}}^{p-1} \cot\left(\pi \frac{z-a}{p}\right) = -\cot\left(\pi \frac{b-a}{p}\right),$$

$$\text{and} \quad \sum_{\substack{z=0 \\ z \neq a,b}}^{p-1} -\cot\left(\pi \frac{z-b}{p}\right) = \cot\left(\pi \frac{a-b}{p}\right).$$

The known equalities

$$1 + \cot\left(\pi \frac{z-a}{p}\right)\cot\left(\pi \frac{z-b}{p}\right)$$
$$= \cot\left(\pi \frac{a-b}{p}\right)\left(\cot\left(\pi \frac{z-a}{p}\right) - \cot\left(\pi \frac{z-b}{p}\right)\right)$$

and

$$\frac{1}{1 - \exp(2\imath\pi t)} = \frac{1}{2} + \frac{\imath}{2}\cot(\pi t)$$

give

$$\left|\sum_{\substack{z=0\\ z\neq a,b}}^{p-1} \frac{1}{1-\exp(2\imath\pi\frac{z-a}{p})}\overline{\left(\frac{1}{1-\exp(2\imath\pi\frac{z-b}{p})}\right)}\right|$$

$$= \left|\sum_{\substack{z=0\\ z\neq a,b}}^{p-1} \left(\frac{1}{2}+\frac{\imath}{2}\cot\left(\pi\frac{z-a}{p}\right)\right)\left(\frac{1}{2}-\frac{\imath}{2}\cot\left(\pi\frac{z-b}{p}\right)\right)\right|$$

$$= \frac{1}{4}\left|\cot\left(\pi\frac{a-b}{p}\right)+\imath\right|\left|\sum_{\substack{z=0\\ z\neq a,b}}^{p-1} \cot\left(\pi\frac{z-a}{p}\right)-\cot\left(\pi\frac{z-b}{p}\right)\right|$$

$$\leqslant C\cot^2\left(\pi\frac{a-b}{p}\right)+C\left|\cot\left(\pi\frac{a-b}{p}\right)\right|$$

$$\leqslant C\sin^{-2}\left(\pi\frac{a-b}{p}\right).$$

Consequently, for an arbitrary function $g : Z_p \to \mathbb{C}$ we have

$$\frac{1}{p}\sum_{z=0}^{p-1}\left|\frac{1}{p}\sum_{\substack{a=0\\ a\neq z}}^{p-1} g(a)\frac{1}{1-\exp(2\pi\imath(z-a)/p)}\right|^2$$

$$\leqslant \frac{1}{p^3}\left|\sum_{a=0}^{p-1}\sum_{\substack{a=0\\ a\neq b}}^{p-1} g(a)\bar{g}(b)\sum_{\substack{z=0\\ z\neq a,b}}^{p-1}\left(\frac{1}{2}+\frac{\imath}{2}\cot\left(\pi\frac{z-a}{p}\right)\right)\left(\frac{1}{2}-\frac{\imath}{2}\cot\left(\pi\frac{z-b}{p}\right)\right)\right|$$

$$+\ \frac{1}{p^3}\sum_{a=0}^{p-1}|g(a)|^2\sum_{\substack{z=0\\ z\neq a}}^{p-1}\left|\frac{1}{2}+\frac{\imath}{2}\cot\left(\pi\frac{z-a}{p}\right)\right|^2$$

$$\leqslant \frac{C}{p^3}\sum_{a=0}^{p-1}\sum_{\substack{b=0\\ b\neq a}}^{p-1}|g(a)||g(b)|\sin^{-2}\left(\pi\frac{a-b}{p}\right)+\frac{C}{p^3}\sum_{a=0}^{p-1}|g(a)|^2\sum_{\substack{z=0\\ z\neq a}}^{p-1}\frac{p^2}{(z-a)^2}$$

$$\leqslant C\frac{1}{p}\sum_{a=0}^{p-1}|g(a)|^2.$$

The last inequality is followed by the Cauchy–Buniakovskii inequality:

$$\frac{1}{p}\sum_{a=0}^{p-1}|g(a)||g(a+j)| \leqslant \frac{1}{p}\left(\sum_{a=0}^{p-1}|g(a)|^2\right)^{\frac{1}{2}}\left(\sum_{a=0}^{p-1}|g(a+j)|^2\right)^{\frac{1}{2}} = \frac{1}{p}\sum_{a=0}^{p-1}|g(a)|^2.$$

Define for $1 \leqslant A \in \mathbb{N}$ the operator $H_{1,A}$ in the following way:

$$H_{1,A}f(y) := \left| M_{A-1} \int_{\bigcup_{x_{A-1}\neq y_{A-1}} I_A(y_0,\dots,y_{A-2},x_{A-1})} f(x) \frac{1}{1-r_{A-1}(y-x)}\, dx \right|$$

$(f \in L^1, y \in G_m)$. By the inequality above we have

$$\begin{aligned}
&\frac{1}{m_{A-1}} \sum_{y_{A-1}=0}^{m_{A-1}-1} |H_{1,A}f(y)|^2 \\
&\quad = \frac{1}{m_{A-1}} \sum_{y_{A-1}=0}^{m_{A-1}-1} \left| \frac{1}{m_{A-1}} \sum_{x_{A-1}\neq y_{A-1}} E_A f(y_0,\dots,y_{A-2},x_{A-1}) \frac{1}{1-r_{A-1}(y-x)} \right|^2 \\
&\quad \leqslant \frac{C}{m_{A-1}} \sum_{x_{A-1}=0}^{m_{A-1}-1} |E_A f(y_0,\dots,y_{A-2},x_{A-1})|^2.
\end{aligned}$$

This immediately gives

$$||H_{1,A}f||_2^2 \leqslant \frac{C}{M_A} \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_{A-1}=0}^{m_{A-1}-1} |E_A f(y_0,\dots,y_{A-2},y_{A-1})|^2 = C||E_A f||_2^2 \leqslant C||f||_2^2.$$

That is, we have proved that the operator $H_{1,A}$ is of type (L^2, L^2). Since

$$\int_{\bigcup_{x_{A-1}\neq y_{A-1}} I_A(y_0,\dots,y_{A-2},x_{A-1})} \cot\left(\pi \frac{y_{A-1}-x_{A-1}}{m_{A-1}}\right) dx = 0,$$

then

$$\begin{aligned}
&H_{1,A}(E_{A-1}f)(y) \\
&\quad = M_{A-1} \int_{\bigcup_{x_{A-1}\neq y_{A-1}} I_A(y_0,\dots,y_{A-2},x_{A-1})} E_{A-1}f(x) \left(\frac{1}{2} + \frac{\imath}{2} \cot\left(\pi \frac{y_{A-1}-x_{A-1}}{m_{A-1}}\right)\right) dx \\
&\quad \leqslant Cf^*(y).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&||H_1 f||_2^2 \\
&\quad \leqslant \left|\left| \sup_{A\in\mathbb{N}} H_{1,A}f \right|\right|_2^2 \\
&\quad \leqslant \left|\left| \sup_{A\in\mathbb{N}} H_{1,A}(E_A f - E_{A-1}f) + Cf^* \right|\right|_2^2 \\
&\quad \leqslant C||f^*||_2^2 + C \left|\left| \sup_{A\in\mathbb{N}} H_{1,A}(E_A f - E_{A-1}f) \right|\right|_2^2
\end{aligned}$$

$$\leqslant C\|f\|_2^2 + C\sum_{A=1}^{\infty}\|H_{1,A}(E_Af - E_{A-1}f)\|_2^2$$
$$\leqslant C\|f\|_2^2 + C\sum_{A=1}^{\infty}\|E_Af - E_{A-1}f\|_2^2$$
$$\leqslant C\|f\|_2^2.$$

This completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *The operator H_1 is of weak type (L^1, L^1).*

Proof. Let $f \in L^1(G_m)$ such that

$$\int_{G_m} f\,d\mu = 0, \quad \operatorname{supp} f \subset \bigcup_{j=\alpha}^{\beta} I_k(z,j) =: I,$$

where $I_k(z,j) = I_{k+1}(z_0, \dots, z_{k-1}, j), z \in G_m$, and $j \in \{\alpha, \alpha+1, \dots, \beta\} \subset \{0, 1, \dots, m_k - 1\}$. Let $\gamma := \lfloor (\alpha+\beta)/2 \rfloor$. Define the distance of $j, k \in \{0, 1, \dots, m_k - 1\} = Z_{m_k}$ as

$$\rho(j,k) := \begin{cases} |j-k| & \text{if } |j-k| \leqslant \dfrac{m_k}{2}, \\ m_k - |j-k| & \text{if } |j-k| > \dfrac{m_k}{2}. \end{cases}$$

In other words, Z_{m_k} is considered as a circle. Define the set $6I$ in the following way:
If $\beta - \alpha + 1 \geqslant m_k/6$, then $6[\alpha, \beta] := \{0, \dots, m_k - 1\}$,

$$6I := \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j) = I_k(z).$$

On the other hand, if $\beta - \alpha + 1 < m_k/6$, then $6[\alpha, \beta] := \{ j \in Z_{m_k} : \rho(j,\gamma) \leqslant 3(\beta - \alpha + 1)\}$,

$$6I := \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j).$$

It is obvious that $\mu(I) \leqslant \mu(6I) \leqslant 6\mu(I)$. Denote by $e_k \in G_m$ the sequence whose kth coordinate is 1, and the rest are zeros ($k \in \mathbb{N}$). For $x \in I$, and $y \in \bigcup_{j \in Z_{m_k} \setminus 6[\alpha,\beta]} I_k(z,j)$ we give an upper bound for

$$\left|\frac{1}{1 - r_k(y-x)} - \frac{1}{1 - r_k(y - \gamma e_k)}\right|,$$

and later for the sum of them. The definition of ρ gives

$$\frac{1}{|\sin(\pi \frac{y_k - x_k}{m_k})|} = \frac{1}{\sin(\pi \frac{\rho(y_k, x_k)}{m_k})} \leqslant C \frac{m_k}{\rho(y_k, x_k)}.$$

Since

$$\frac{1}{1-\exp(2\pi\imath z)}=\frac{1}{2}+\frac{\imath}{2}\cot(\pi z),$$

then we have

$$\begin{aligned}&\left|\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right|\\&\quad=\frac{1}{2}\left\{\frac{\cos(\pi\frac{y_k-x_k}{m_k})}{\sin(\pi\frac{y_k-x_k}{m_k})}-\frac{\cos(\pi\frac{y_k-\gamma}{m_k})}{\sin(\pi\frac{y_k-\gamma}{m_k})}\right\}\\&\quad=\frac{1}{2}\left|\frac{\sin(\pi\frac{x_k-\gamma}{m_k})}{\sin(\pi\frac{y_k-x_k}{m_k})\sin(\pi\frac{y_k-\gamma}{m_k})}\right|\\&\quad\leqslant C\frac{(\beta-\alpha+1)/m_k}{|\sin(\pi\frac{y_k-x_k}{m_k})\sin(\pi\frac{y_k-\gamma}{m_k})|}\\&\quad\leqslant C\frac{(\beta-\alpha+1)m_k}{\rho(y_k,x_k)\rho(y_k,\gamma)}\\&\quad\leqslant C\frac{(\beta-\alpha+1)m_k}{\rho^2(y_k,\gamma)}.\end{aligned}$$

The last inequality is implied by the definition of ρ, $y_k\notin 6[\alpha,\beta]$, and $\rho(y_k,x_k)\geqslant\rho(y_k,\gamma)-(\beta-\alpha+1)\geqslant\frac{2}{3}\rho(y_k,\gamma)$. Consequently, we have

$$\begin{aligned}&\frac{1}{m_k}\sum_{\substack{y_k\in\{0,\dots,m_k-1\}\\ y_k\notin 6[\alpha,\beta]}}\left|\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right|\\&\quad\leqslant C\sum_{\{y_k:\rho(y_k,\gamma)>3(\beta-\alpha+1)\}}\frac{\beta-\alpha+1}{\rho^2(y_k,\gamma)}\leqslant C.\end{aligned}$$

In the sequel, we consider

$$H_{1,A}f(y)=\left|M_{A-1}\int_{\bigcup_{x_{A-1}\neq y_{A-1}}I_A(y_0,\dots,y_{A-2},x_{A-1})}f(x)\frac{1}{1-r_{A-1}(y-x)}dx\right|,$$

where $y\in G_m\backslash 6I$. This means, that either there exists an $i\leqslant k-1$, such that $y_i\neq z_i$, or $y_0=z_0,\dots,y_{k-1}=z_{k-1}$, and $y_k\notin 6[\alpha,\beta]$.

The case $A>k+1$. In this case

$$\bigcup_{x_{A-1}\neq y_{A-1}}I_A(y_0,\dots,y_{A-2},x_{A-1})\subset I_{A-1}(y_0,\dots,y_{A-2})\subset I_{k+1}(y_0,\dots,y_k).$$

If there exists a $i\leqslant k-1$, such that $y_i\neq z_i$, then the sets $I_k(z_0,\dots,z_{k-1})\supset I$, and $I_{k+1}(y_0,\dots,y_k)$ are disjoint. Consequently, $H_{1,A}f(y)=0$.

On the other hand, if $y_0=z_0,\dots,y_{k-1}=z_{k-1}$, and $y_k\notin 6[\alpha,\beta]$, then the intervals $I_{k+1}(y_0,\dots,y_k)=I_{k+1}(z_0,\dots,z_{k-1},y_k)$, and $I=\bigcup_{j=\alpha}^{\beta}I_{k+1}(z_0,\dots,z_{k-1},j)$ are disjoint. Anyway, we have $H_{1,A}f(y)=0$.

The case $A<k+1$. That is, $A-1\leqslant k-1$. If

$$I_A(y_0,\dots,y_{A-2},x_{A-1})\cap I\neq\emptyset,$$

then the condition $y_0=z_0,\dots,y_{A-2}=z_{A-2},x_{A-1}=z_{A-1}$ must be fulfilled. It follows that $I\subset I_k(z_0,\dots,z_{k-1})\subset I_A(y_0,\dots,y_{A-2},x_{A-1})$. Thus, $I_A(y_0,\dots,y_{A-2},x_{A-1})\cap I=I$. Consequently, the function $r_{A-1}(y-x)$ is constant as x ranges over I. This gives

$$\begin{aligned}H_{1,A}f(y)&=\left|M_{A-1}\int_I f(x)\frac{1}{1-r_{A-1}(y-x)}\,dx\right|\\&=\left|M_{A-1}\frac{1}{1-r_{A-1}(y-z_{A-1}e_{A-1})}\int_I f(x)\,dx\right|=0.\end{aligned}$$

Consequently, $H_{1,A}f(y)$ may differ from zero only in the case $A=k+1$. It follows

$$H_1f(y)=\left|M_k\int_{\bigcup_{x_k\neq y_k}I_{k+1}(y_0,\dots,y_{k-1},x_k)}f(x)\frac{1}{1-r_k(y-x)}\,dx\right|.$$

Recall that $y\in G_m\backslash 6I$. Moreover, if $H_{1,A}f(y)\neq 0$, then $y_0=z_0,\dots,y_{k-1}=z_{k-1}$, and $y_k\notin 6[\alpha,\beta]$. These assumptions give

$$\begin{aligned}&\int_{G_m\backslash 6I}|H_1f(y)|\,dy\\&=\frac{1}{M_{k+1}}\sum_{\substack{y_k=0,\dots,m_k-1\\ y_k\notin 6[\alpha,\beta]}}\left|M_k\int_{\bigcup_{x_k\in[\alpha,\beta]}I_{k+1}(z_0,\dots,z_{k-1},x_k)}f(x)\frac{1}{1-r_k(y-x)}\,dx\right|\\&=\frac{1}{M_{k+1}}\sum_{\substack{y_k=0,\dots,m_k-1\\ y_k\notin 6[\alpha,\beta]}}\left|M_k\int_{\bigcup_{x_k\in[\alpha,\beta]}I_{k+1}(z_0,\dots,z_{k-1},x_k)}f(x)\right.\\&\quad\left.\times\left(\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right)dx\right|\\&\leqslant\int_{\bigcup_{x_k\in[\alpha,\beta]}I_{k+1}(z_0,\dots,z_{k-1},x_k)}|f(x)|\,dx\\&\quad\cdot\frac{1}{m_k}\sum_{\substack{y_k=0,\dots,m_k-1\\ y_k\notin 6[\alpha,\beta]}}\left|\frac{1}{1-r_k(y-x)}-\frac{1}{1-r_k(y-\gamma e_k)}\right|\\&\leqslant C\int_{\bigcup_{x_k\in[\alpha,\beta]}I_{k+1}(z_0,\dots,z_{k-1},x_k)}|f(x)|\,dx=C\|f\|_1.\end{aligned}$$

After then, let $f\in L^1(G_m)$, and $\lambda>\|f\|_1>0$ arbitrary. Apply the Calderon–Zygmund decomposition lemma (Lemma 2.2). That is,

$$f=f_0+\sum_{j=1}^{\infty}f_j,\quad \|f_0\|_\infty\leqslant C\lambda,\quad \|f_0\|_1\leqslant C\|f\|_1,$$

$$\operatorname{supp} f_j \subset \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l) = J_j, \quad \int_{G_m} f_j \, d\mu = 0 \quad (j \in \mathbb{P}),$$

$$F = \bigcup_{j \in \mathbb{P}} J_j, \quad \mu(F) \leqslant C \frac{||f||_1}{\lambda},$$

and the sets J_j are disjoint $(j \in \mathbb{P})$. This, and the fact that the operator H_1 is of type (L^2, L^2) imply that

$$\mu(H_1 f_0 > \lambda) \leqslant \frac{||H_1 f_0||_2^2}{\lambda^2} \leqslant C \frac{||f_0||_2^2}{\lambda^2} \leqslant C \frac{||f_0||_1}{\lambda} \leqslant C \frac{||f||_1}{\lambda}.$$

Let

$$6F \coloneqq \bigcup_{j \in \mathbb{P}} 6J_j.$$

It is obvious that $\mu(6F) \leqslant 6\mu(F) \leqslant C||f||_1/\lambda$. Consequently, since the operator H_1 is sublinear, then we have

$$\begin{aligned}
&\mu(H_1 f > 2\lambda) \\
&\quad \leqslant \mu(H_1 f_0 > \lambda) + \mu\left(H_1\left(\sum_{j=1}^{\infty} f_j\right) > \lambda\right) \\
&\quad \leqslant C \frac{||f||_1}{\lambda} + \mu(6F) + \mu\left(\left\{y \notin 6F : \sum_{j=1}^{\infty} H_1 f_j(y) > \lambda\right\}\right) \\
&\quad \leqslant C \frac{||f||_1}{\lambda} + \frac{1}{\lambda} \int_{G_m \setminus 6F} \sum_{j=1}^{\infty} H_1 f_j(y) \, dy \\
&\quad \leqslant C \frac{||f||_1}{\lambda} + C \frac{1}{\lambda} \sum_{j=1}^{\infty} ||f_j||_1 \\
&\quad \leqslant C \frac{||f||_1}{\lambda}.
\end{aligned}$$

This implies that the operator H_1 is of weak type (L^1, L^1). This completes the proof of Lemma 2.4. □

For any $1 \leqslant j \in \mathbb{N}$ define the operator H_j in the following way:

$$H_j f(y) \coloneqq \sup_{j \leqslant A \in \mathbb{N}} \left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0, \ldots, y_{A-j-1}, x_{A-j}, \ldots, y_{A-1})} f(x) \frac{1}{1 - r_{A-j}(y - x)} \, dx \right|,$$

where $y \in G_m$. In Lemma 2.4 we proved that the operator H_1 is of weak type (L^1, L^1). This will be generalized for the operators H_j, where $j \geqslant 2$. More exactly, we not only prove that operators H_j are of weak type (L^1, L^1), uniformly in j, we prove even more. This will play a fundamental role in the proof of Theorem 2.1.

Lemma 2.5. *There exists an absolute constant* $C>0$ *such that for all* $j\in\mathbb{N}, f\in L^1(G_m)$, *and* $\lambda>0$

$$\mu(H_jf>\lambda)\leqslant C\frac{j^2\|f\|_1}{2^j\lambda}.$$

Proof. For $j=1$ this lemma is nothing else but Lemma 2.4. In the proof of Lemma 2.5 we will use Lemma 2.4. Apply a permutation for the coordinate groups of the Vilenkin group G_m such that for all $A\geqslant j, A\in\mathbb{N}$ the $(A-j)$th coordinate group and the $(A-1)$th coordinate group will be adjacent. Then Lemma 2.4 can be applied for the modified Vilenkin group, and this will verify Lemma 2.5:

$$\begin{aligned} &H_jf(y)\\ &\leqslant\sum_{k=0}^{j-1}\sup_{\substack{j\leqslant A\in\mathbb{N}\\ A\equiv k \bmod j}}\left|M_{A-j}\int_{\bigcup_{x_{A-j}\neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x)\frac{1}{1-r_{A-j}(y-x)}\,dx\right|\\ &=:\sum_{k=0}^{j-1} H_{j,k}f(y). \end{aligned}$$

We prove the existence an absolute constant $C>0$ such that for all $\lambda>0,\ f\in L^1(G_m)$, and j,k the inequality

$$\mu(H_{j,k}f>\lambda)\leqslant C\frac{\|f\|_1}{2^j\lambda}$$

holds. This inequality immediately gives

$$\begin{aligned} &\mu(H_jf(y)>\lambda)\\ &\leqslant\mu\left(\bigcup_{k=0}^{j-1}\left\{H_{j,k}f>\frac{\lambda}{j}\right\}\right)\\ &\leqslant\sum_{k=0}^{j-1}\mu\left(H_{j,k}f>\frac{\lambda}{j}\right)\\ &\leqslant C\|f\|_1\frac{j^2}{2^j\lambda}. \end{aligned}$$

This completes the proof of Lemma 2.5. Let

$$\begin{aligned} H_{j,k}^Nf(y)\coloneqq\sup\Bigg\{\Bigg|M_{A-j}\int_{\bigcup_{x_{A-j}\neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x)\\ \times\frac{1}{1-r_{A-j}(y-x)}\,dx\Bigg| : j\leqslant A\leqslant Nj+k, A\equiv k \bmod j\Bigg\}. \end{aligned}$$

Since $H_{j,k}^Nf$ is monotone increasing as N gets larger, then by measure theory if we prove that the operators $2^jH_{j,k}^N$ are of weak type (L^1,L^1), uniformly in N (it means that the constant C does not depend on N,j,k), then $2^jH_{j,k}$ is also of weak type

(L^1, L^1). This would imply

$$\mu(H_{j,k}f > \lambda) = \mu(2^j H_{j,k}f > 2^j\lambda) \leqslant C\frac{||f||_1}{2^j\lambda}.$$

Recall that the Vilenkin group G_m is the complete direct product of its coordinate groups Z_{m_l}, that is, $G_m = \times_{l=0}^{\infty} Z_{m_l}$. We define another Vilenkin group. Its coordinate groups will be the same, but with certain rearrangement. Let the function $\alpha : \mathbb{N} \to \mathbb{N}$ be defined in the following way. If $n \geqslant k + Nj$, or $n \not\equiv k, k-1 \bmod j$, then

$$\alpha(n) := n$$

and

$$\alpha(k + lj) := k + (l+1)j - 1, \quad \alpha(k + (l+1)j - 1) := k + lj,$$

for all $l < N, l \in \mathbb{N}$. Then define the Vilenkin group $G_m^{j,k}$ as:

$$G_m^{j,k} = \mathop{\times}_{l=0}^{\infty} Z_{m_{\alpha(l)}}.$$

We give a measure preserving bijection between the two Vilenkin groups. We denote it by α, or more precisely (if it is needed) by $\alpha_{j,k}$. It will not cause any confusion. That is,

$$\alpha = \alpha_{j,k} : G_m \to G_m^{j,k},$$

and let the nth coordinate of the sequence $\alpha_{j,k}(x) \in G_m^{j,k}$ be $x_{\alpha(n)}$. That is,

$$(\alpha(x))_n = x_{\alpha(n)} \quad (n \in \mathbb{N}).$$

Consequently, we have a finite permutation of the coordinates. This is very important for us, since when we discuss the operator H_1 on the Vilenkin group $G_m^{j,k}$, then we can apply the result given (H_1 is of weak type (L^1, L^1)) for the operator $H_{j,k}^N$ on the Vilenkin group G_m.

Denote by $\tilde{m}$ the sequence for which $\tilde{m}_l = m_{\alpha(l)}$. Introduce the notations $\tilde{x} := \alpha(x)(x \in G_m)$, $\tilde{r}_l := r_{\alpha(l)}(l \in \mathbb{N})$. Recall that $A \equiv k \bmod j$. Then we have

$$\begin{aligned}
1 - r_{A-j}(y - x) &= 1 - \exp\left(2\pi\imath\frac{y_{A-j} - x_{A-j}}{m_{A-j}}\right) \\
&= 1 - \exp\left(2\pi\imath\frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{m_{A-j}}\right) \\
&= 1 - \exp\left(2\pi\imath\frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{\tilde{m}_{A-1}}\right) \\
&= 1 - \tilde{r}_{A-1}(\tilde{y} - \tilde{x}).
\end{aligned}$$

Moreover, denote by $\tilde{M}$ the sequence of the generalized powers with respect to the sequence $\tilde{m}$. This gives

$$\begin{aligned}
&\tilde{m}_0 \ldots \tilde{m}_{A-2} \\
&\quad = \tilde{M}_{A-1}
\end{aligned}$$

$$
\begin{aligned}
&= m_0 m_1 \dots m_{A-j-1} m_{A-j+1} \dots m_{A-1} \\
&= \frac{m_0 \dots m_{A-1}}{m_{A-j}} \\
&= M_{A-j} \frac{m_{A-j} m_{A-j+1} \dots m_{A-1}}{m_{A-j}} \\
&= M_{A-j} m_{A-j+1} \dots m_{A-1}.
\end{aligned}
$$

This gives $M_{A-j} \leqslant \tilde{M}_{A-1}/2^{j-1}$. By the above written we get

$$
\begin{aligned}
&\left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \frac{1}{1 - r_{A-j}(y-x)}\, dx \right| \\
&= \left| M_{A-j} \int_{\bigcup_{\tilde{x}_{A-1} \neq \tilde{y}_{A-1}} I_A(\tilde{y}_0,\dots,\tilde{y}_{A-2},\tilde{x}_{A-1})} \tilde{f}(\tilde{x}) \frac{1}{1 - \tilde{r}_{A-1}(\tilde{y}-\tilde{x})}\, d\tilde{x} \right| \\
&\leqslant \frac{1}{2^{j-1}} H_1 \tilde{f}(\tilde{y}),
\end{aligned}
$$

where the function $\tilde{f}$ is defined on $G_m^{j,k}$ by $f(x) = \tilde{f}(\tilde{x})$ for all $x \in G_m$. The definition of $H_{j,k}^N$ gives

$$
H_{j,k}^N f(y) \leqslant \frac{1}{2^{j-1}} H_1 \tilde{f}(\tilde{y}).
$$

By Lemma 2.4 it follows

$$
\begin{aligned}
&\mu(\{y \in G_m : H_{j,k}^N f(y) > \lambda\}) \\
&\quad \leqslant \mu(\{\tilde{y} \in G_m^{j,k} : (H_1 \tilde{f})(\tilde{y}) > \lambda 2^{j-1}\}) \\
&\quad \leqslant C \frac{\|\tilde{f}\|_1}{\lambda 2^j} \\
&\quad = C \frac{\|f\|_1}{\lambda 2^j}.
\end{aligned}
$$

The operator $H_{j,k}^N$ is monotone increasing in N. Consequently, we have

$$
\begin{aligned}
&\mu(H_{j,k} f > \lambda) \\
&\quad = \mu\left(\bigcup_{n \in \mathbb{N}} \{H_{j,k}^N f > \lambda\} \right) \\
&\quad = \lim_{N \to \infty} \mu(H_{j,k}^N f > \lambda) \\
&\quad \leqslant C \frac{\|f\|_1}{\lambda 2^j}.
\end{aligned}
$$

This, as we said before, gives the inequality $\mu(H_j f > \lambda) \leqslant C \frac{j^2 \|f\|_1}{2^j \lambda}$. That is, the proof of Lemma 2.5 is complete. $\square$

The last lemma we need in order to prove Theorem 2.1.

Lemma 2.6. *Let* $A > t$, $t, A \in \mathbb{N}$, $z \in I_t \backslash I_{t+1}$. *Then*

$$K_{M_A}(z) = \begin{cases} 0 & \text{if } z - z_t e_t \notin I_A, \\ \dfrac{M_t}{1 - r_t(z)} & \text{if } z - z_t e_t \in I_A. \end{cases}$$

Proof. Observe that

$$\begin{aligned} & M_A K_{M_A}(z) \\ & = \sum_{k=1}^{M_A} D_k(z) \\ & = \sum_{k=1}^{M_A} \psi_k(z) \left(\sum_{j=0}^{t-1} k_j M_j + M_t \sum_{i=m_t-k_t}^{m_t-1} r_t^i(z) \right) \\ & =: J_1 + J_2. \end{aligned}$$

We remind the reader that $k = \sum k_j M_j$, and $D_{M_A}(z) = 0$. First, we prove that $J_1 = 0$.

$$J_1 = \sum_{k_0=0}^{m_0-1} \cdots \sum_{k_{t-1}=0}^{m_{t-1}-1} \sum_{k_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{k_{A-1}=0}^{m_{A-1}-1} \left(\prod_{\substack{l=0 \\ l \neq t}}^{A-1} r_l^{k_l}(z) \right) \sum_{j=0}^{t-1} k_j M_j \sum_{k_t=0}^{m_t-1} r_t^{k_t}(z) = 0,$$

because $\sum_{k_t=0}^{m_t-1} r_t^{k_t}(z) = 0$. We recall that $z \in I_t \backslash I_{t+1}$. Since

$$\sum_{k_0=0}^{m_0-1} \cdots \sum_{k_{t-1}=0}^{m_{t-1}-1} \sum_{k_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{k_{A-1}=0}^{m_{A-1}-1} \left(\prod_{\substack{l=0 \\ l \neq t}}^{A-1} r_l^{k_l}(z) \right) = \prod_{\substack{l=0 \\ l \neq t}}^{A-1} \left(\sum_{k_l=0}^{m_l-1} r_l^{k_l}(z) \right)$$

then if $z - z_t e_t \notin I_A$, then we also have $J_2 = 0$. That is, $K_{M_A}(z) = 0$ in this case. On the other hand, if $z - z_t e_t \in I_A$, then

$$\begin{aligned} & M_A K_{M_A}(z) \\ & = M_t \prod_{\substack{l=0 \\ l \neq t}}^{A-1} m_l \sum_{k_t=0}^{m_t-1} r_t^{k_t}(z) \sum_{i=m_t-k_t}^{m_t-1} r_t^i(z) \\ & = M_t \frac{M_A}{m_t} \sum_{k_t=0}^{m_t-1} r_t^{k_t}(z) \sum_{i=m_t-k_t}^{m_t-1} r_t^i(z) \\ & = \frac{M_t M_A}{m_t} \sum_{k_t=0}^{m_t-1} \frac{r_t^{k_t}(z) - 1}{r_t(z) - 1}. \end{aligned}$$

Since

$$\sum_{k_t=0}^{m_t-1} \frac{r_t^{k_t}(z)}{r_t(z)-1} = 0,$$

then we have

$$M_A K_{M_A}(z) = M_t M_A \frac{1}{1-r_t(z)}.$$

This completes the proof of Lemma 2.6. □

Proof of Theorem 2.1. With some easy calculations we get

$$\begin{aligned}
&|\sigma_{M_A} f(y)| \\
&\quad = \left| \int_{G_m} f(x) K_{M_A}(y-x)\, dx \right| \\
&\quad \leqslant \left| \int_{I_A(y)} f(x) K_{M_A}(y-x)\, dx \right| + \left| \sum_{t=0}^{A-1} \int_{I_t(y) \setminus I_{t+1}(y)} f(x) K_{M_A}(y-x)\, dx \right| \\
&\quad \leqslant M_A \int_{I_A(y)} |f(x)|\, dx \\
&\qquad + \sum_{t=0}^{A-1} \left| M_t \int_{\bigcup_{x_t \neq y_t} I_A(y_0,\dots,y_{t-1},x_t,y_{t+1}\dots,y_{A-1})} f(x) \frac{1}{1-r_t(y-x)}\, dx \right| \\
&\quad \leqslant |f|^*(y) + \sum_{j=1}^{\infty} H_j f(y).
\end{aligned}$$

We recall that for $z \in I_A$ we have $K_{M_A}(z) = K_{M_A}(0) = \frac{1}{M_A} \sum_{k=0}^{M_A-1} k = \frac{M_A-1}{2}$. That is, for the maximal operator $\sigma^* := \sup_A |\sigma_{M_A}|$ we have

$$\sigma^* f \leqslant |f|^* + \sum_{j=1}^{\infty} H_j f.$$

This by Lemma 2.5 immediately gives for all $f \in L^1(G_m)$, and $\lambda > 0$:

$$\begin{aligned}
\mu(\sigma^* f > \lambda) &\leqslant \mu\left(|f|^* > \frac{\lambda}{2} \right) + \mu\left(\sum_{j=1}^{\infty} H_j f > \frac{\lambda}{2} \right) \\
&\leqslant \frac{C\|f\|_1}{\lambda} + \mu\left(\bigcup_{j=1}^{\infty} \left\{ H_j f > \frac{\lambda}{4j^2} \right\} \right) \\
&\leqslant \frac{C\|f\|_1}{\lambda} + \sum_{j=1}^{\infty} \mu\left(H_j f > \frac{\lambda}{4j^2} \right)
\end{aligned}$$

$$\leqslant \frac{C\|f\|_1}{\lambda} + \sum_{j=1}^{\infty} \frac{Cj^4\|f\|_1}{2^j\lambda}$$
$$\leqslant \frac{C\|f\|_1}{\lambda}.$$

This proves that the operator σ^* is of weak type (L^1, L^1). Finally, let $f \in L^1(G_m)$, and $\varepsilon > 0$. Since the set of Vilenkin polynomials (the finite linear combinations of Vilenkin functions) is dense in the space $L^1(G_m)$, then for each $\delta > 0$ there exists a Vilenkin polynomial P, such that $\|f - P\|_1 < \delta$. Since the relation

$$\lim_{A\to\infty} \sigma_{M_A} P = P$$

holds everywhere, then we have

$$\mu(\limsup|\sigma_{M_A} f - f| > \varepsilon)$$
$$\leqslant \mu\left(\limsup|\sigma_{M_A}(f - P)| > \frac{\varepsilon}{3}\right) + \mu\left(|P - f| > \frac{\varepsilon}{3}\right)$$
$$\leqslant \frac{C}{\varepsilon}\|f - P\|_1$$
$$\leqslant \frac{C\delta}{\varepsilon}.$$

Since $\delta > 0$ is arbitrary, then it follows

$$\mu(\limsup|\sigma_{M_A} f - f| > \varepsilon) = 0$$

for all $\varepsilon > 0$. Consequently,

$$\mu(\limsup|\sigma_{M_A} f - f| > 0) = 0,$$

that is, $\sigma_{M_A} f \to f$ almost everywhere. The proof of Theorem 2.1 is complete. □

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